Directed and undirected multiurn models in a one-dimensional ring

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The flea model by Ehrenfest describes the jumps of a fixed number of fleas between two dogs. In each time step a randomly selected flea jumps on the other dog. We study directed and undirected multiurn models in a one-dimensional ring. The introduced models represent generalizations of three recently proposed multiurn models which themselves are generalizations of Ehrenfest's model. The models are solved analytically. For the directed case we find oscillations of the average number of balls or fleas in a certain urn before the system reaches its equilibrium state. The discussed models may serve as basic models of dynamics of granular media in connected periodic compartment systems.

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INTRODUCTION

The famous Ehrenfest model describes the process of equilibration together with thermal fluctuations around the stationary state [1,2]. A total number of N fleas sit on two dogs. In a time step a randomly selected flea jumps from one dog to the other. In Ref. [3] we introduced a generalized model where each flea jumps with a given probability p. Kao and Luan recently introduced another generalization of Ehrenfest's model [4]. The authors arrange M dogs in a circle where repeatedly a randomly chosen flea hops to the right side neighbor dog. In a succeeding model a randomly picked flea hops from dog j to dog $j+k$ with probability p_k, $k=1,\dots,M$ [5]. As common features of both models, first, only directed jumps are allowed, and second, only one flea can hop per time step. These rather restrictive assumptions had the tradeoff to yield relatively simple solutions. Here we demonstrate that an analytical solution is still attainable in the more general case where all fleas can jump in a time step. This generalization applies for systems in which fleas, balls, or granular particles, can simultaneously change their dog, box, or container compartment, respectively. Particle dynamics in vibrating containers consisting of, say, M (connected) compartments, where a particle can hop from one compartment to another, has been extensively discussed in the literature; see Refs. [6,7] and references therein. Consequently, the models we discuss in this article may serve as basic models for granular dynamics of particles in container compartments being relevant in a variety of technological and industrial processes [8,9].

I. NEXT NEIGHBOR MODEL

We start with an undirected nearest neighbor model. Suppose that M dogs are harassed by N (distinguishable) fleas where the dogs form a circle (see Fig. 1). Let $\mathbf{n}^t = (n_1^t, n_2^t, \dots, n_M^t)$ be the state vector, i.e., n_d^t denotes the number of fleas on dog $d=1,2,\dots,M$ at time step $t=0, 1,\dots$. At a time step t a flea jumps with probability p to the left, or with probability p to the right next neighbor dog, or remains on its victim with probability $1-2p$. Let $P_d^t(n)$ denote the probability to find n fleas on dog d at time t. We now derive $P_d^t(n)$ returning to the picture of N uncoupled one-flea systems.

II. ONE-FLEA SYSTEMS

Let $Q_f^t(d)$ denote the probability to find flea $f=1,\dots,N$ at time t on dog d when the flea initially was on dog d_f^0. To find $Q_f^t(d)$, we use the method of Markov chains. Let A_p be the symmetric $M \times M$ Markov matrix

$$(A_p)_j^k = (1 - 2p)\delta_{j,k} + p(\delta_{j,k-1} + \delta_{j,k+1}) \qquad (1)$$

where $\delta_{j,k}$ is Kronecker's symbol and periodic boundary conditions are assumed, that is, indices $j<1$ are identified with $j+M$, and indices $j>M$ with $j-M$, respectively. Let λ_j denote the jth eigenvalue of A_p and x_j^k denote the kth component of the jth eigenvector of A_p where $j,k=1,\dots,M$. Further, let Λ denote the diagonalized matrix to the decomposition $A_p = X\Lambda X^T$ with $X \equiv (x_j^k)$, $X^T X = \mathbb{1}$. The eigenvalues of A_p can be expressed as

$$\lambda_{j,p} = 1 + 2p\left[\cos\left(\frac{2\pi j}{M}\right) - 1\right], \qquad (2)$$

and the corresponding eigenvectors are

$$x_j^k = \frac{1}{\sqrt{M}}\left[\cos\left(\frac{2\pi jk}{M}\right) + \sin\left(\frac{2\pi jk}{M}\right)\right]. \qquad (3)$$

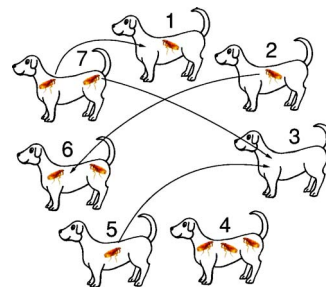As a consequence, we obtain the single-flea probability distribution

FIG. 1. (Color online) Schematic of the model for $N=10$ fleas on $M=7$ dogs. Depending on the model version discussed, in a time step a flea can jump to the next neighbor dog, or to the kth neighbor dog.

$$Q^t_{f,p}(d) = \sum_{j=1}^{N} \lambda^t_{j,p} x^d_j x^j_{d^0_f}. \qquad (4)$$

Equipped with Eq. (4), we return to N fleas. Assuming n^0_d fleas on dog d at time $t=0$, $P^t_d(n)$ can be readily derived. We simply have to add up all possibilities to end up with n fleas at time t when starting with n^0_d. To archive this, we have to sum over all combinations $b_j=0,1$, $j=1,\ldots,N$, that represent a total number of n fleas, i.e., $\Sigma b_j = n$:

$$P^t_d(n) = \sum_{\substack{N \\ \sum_{i=1} b_i = n, \\ b_j = 0,1}} \prod_{f=1}^{N} Q^t_{f,p}(d)^{b_j}[1 - Q^t_{f,p}(d)]^{1-b_j}. \qquad (5)$$

The powers in Eq. (5) are either 0 or 1, depending on whether flea j contributes to the total number of n fleas or not. As a consequence, for $b_j=1$ the probability $Q^t_{f,p}(d)$ is summed up whereas the probability $1 - Q^t_{f,p}(d)$ contributes for $b_j=0$ exclusively. Alternatively, the powers can be eliminated by writing

$$P^t_d(n) = \sum_{\substack{N \\ \sum_{i=1} b_i = n, \\ b_j = 0,1}} \prod_{f=1}^{N} |1 - b_j - Q^t_{f,p}(d)| \qquad (6)$$

which is a simpler expression.

III. ALL FLEAS ON THE FIRST DOG

When all N fleas sit at time $t=0$ on, say, the first dog ($d=1$), Eqs. (5) and (6) are greatly simplified to

$$P^t_d(n) = \binom{N}{n} Q^t_{1,p}(d)^n [1 - Q^t_{1,p}(d)]^{N-n} \overset{N \gg 1}{\simeq} \frac{q^n e^{-q}}{n!} \qquad (7)$$

where $q \equiv N Q^t_{1,p}(d)$. Consequently, the average number of fleas on dog d is given by

$$\langle n_d \rangle_t = Q^t_{1,p}(d) N \sum_{n=1}^{N} n \binom{N}{n} [1 - Q^t_{1,p}(d)]^{N-n} \qquad (8)$$

$$\overset{N \gg 1}{\simeq} q\left(1 - \frac{q^N e^{-q}}{N!}\right). \qquad (9)$$

Figure 2 shows the fraction of the total number of fleas when initially all fleas are placed on dog $d=1$ for various values of the jump rate p. The system relaxes to its equilibrium state $\mathbf{n}^t \rightarrow (N/M,\ldots,N/M)$ for $t \rightarrow \infty$ the faster the higher is the jump rate p. Figure 3 shows the corresponding log-log plot for different numbers of fleas N. Due to the independence of the fleas the graphs of the average number of fleas on dog $d=1$ merge into one line even for different values of N. As expected, the slope α of the line is approximately $\alpha=1/2$ resulting from the time evolution of the distribution's peak height of a random walker which is determined by the reciprocal variance: $1/\sigma(t) \sim t^{-1/2}$ [3,10].
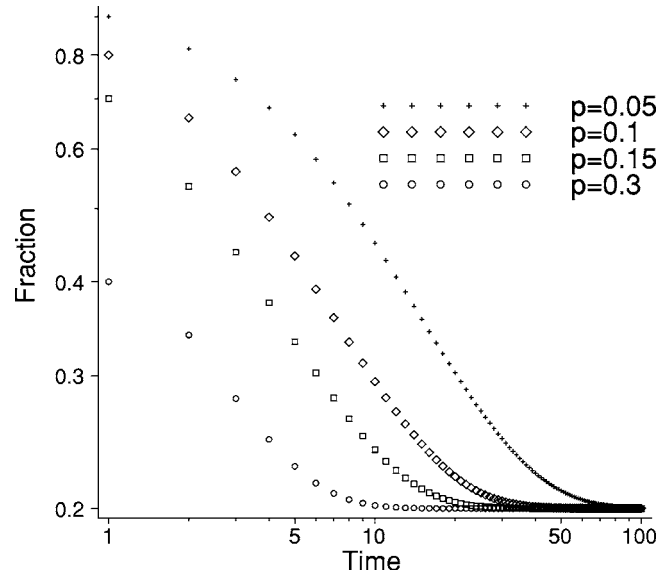

FIG. 2. Fraction of the average number of fleas on dog $d=1$ as a function of time assuming all fleas on dog $d=1$ at time $t=0$. The next neighbor jump rate p is varied for the case of $N=50$ fleas on $M=5$ dogs.

IV. GENERAL UNDIRECTED MODEL

We now discuss the case where the fleas are allowed to jump to the dogs they want to. Let p_k denote the probability to jump from the jth to the $(j+k)$th dog whereas dog $M+1$ is identified with dog 1. For M even the normalization condition reads
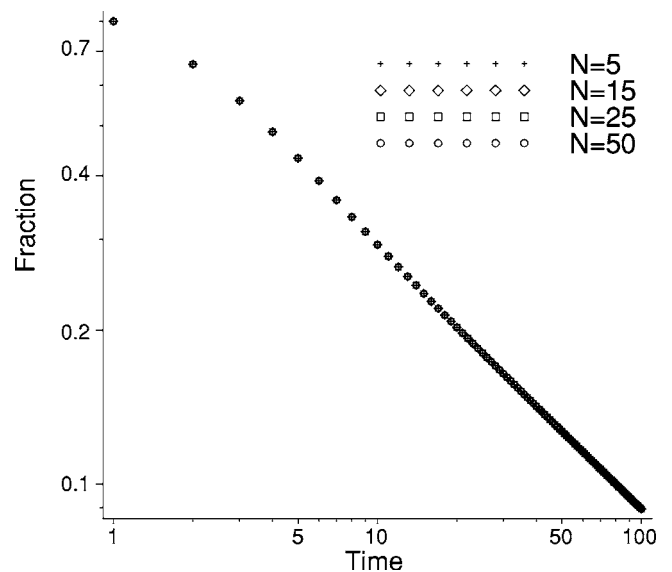

FIG. 3. Log-log plot of the fraction of the average number of fleas on dog $d=1$ versus time for $N=5,15,25,50$ fleas on $M=30$ dogs. Initially all fleas were placed on dog $d=1$. Here, only jumps to the next neighbor dogs with probability $p=1/3$ are allowed. A flea stays on his host with probability $1-2p=2/3$.

$$p_0 + 2 \sum_{k=1}^{M/2-1} p_k + p_{M/2} = 1; \qquad (10)$$

for M odd

$$p_0 + 2 \sum_{k=1}^{(M-1)/2} p_k = 1 \qquad (11)$$

holds. For simplicity, in this section let us restrict ourselves to M even. For this case, the Markov matrix reads

$$(A_{\mathbf{p}})_j^k = p_0 \delta_{j,k} + \sum_{l=0}^{M/2} p_l(\delta_{j,k-l} + \delta_{j,k+l}). \qquad (12)$$

Due to its symmetry the eigenvectors of $A_{\mathbf{p}}$ are also given by Eq. (3). The corresponding eigenvalues differ from Eq. (2), being

$$\lambda_{j,\mathbf{p}} = p_0 + 2 \sum_{k=1}^{M/2-1} p_k \cos\left(\frac{2\pi jk}{M}\right) + p_{M/2}(-1)^k, \qquad (13)$$

where $\mathbf{p}$ indicates the dependence on the set of probabilities p_k, $k = 0, \dots, M/2$. We obtain therefore the one-flea probability distribution

$$Q_{f,\mathbf{p}}^t(d) = \sum_{j=1}^{N} \lambda_{j,\mathbf{p}}^t x_j^d x_{d_f^0}^j. \qquad (14)$$

For the special case $p_1 = p_{M-1} = p$, $p_k = 0$, $k = 2, \dots, M-2$ one obtains the next neighbor model discussed in the previous sections.

V. DIRECTED TRANSITIONS

So far we considered undirected state transitions, i.e., the fleas made no difference between left and right. In this paragraph we investigate the simplest possible model for directed next neighbor transitions. A flea jumps with probability p from dog d to next dog $d+1$ and remains on its host with $1-p$. Hence, the corresponding transition matrix B_p is asymmetric:

$$(B_p)_j^k = (1-p)\delta_{j,k} + p\,\delta_{j,k-1}. \qquad (15)$$

Its eigenvalues are

$$\mu_{j,p} = 1 + p\left[\exp\left(-\frac{2\pi ij}{M}\right) - 1\right], \qquad (16)$$

where $i^2 = -1$ (as throughout the article in the argument of exponentials). The corresponding eigenvectors are

$$y_j^k = \frac{1}{\sqrt{M}}\exp\left(\frac{2\pi ijk}{M}\right). \qquad (17)$$

Thus, we finally obtain the one-flea distribution

$$\tilde{Q}_{f,p}^t(d) = \sum_{j=1}^{N} \mu_{j,p}^t y_j^d y_{d_f^0}^{j*}, \qquad (18)$$

where the asterisk represents the imaginary conjugate, and as expected, the imaginary eigenvectors combine to real prob-
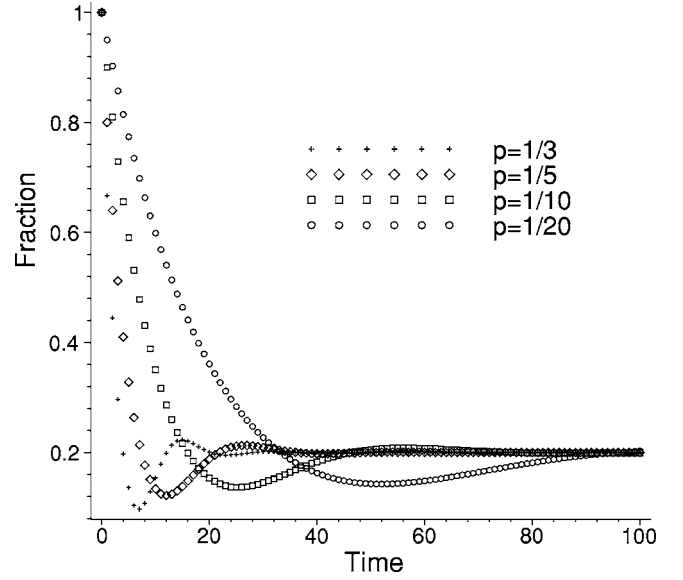

FIG. 4. Average number of fleas on dog $d=1$ as a function of time for $N=50$ fleas on $M=5$ dogs. The fraction, i.e., $\langle n \rangle_t / N$, is displayed for the case that all fleas were at the beginning on dog $d=1$. Here, only jumps to the right neighbor dog with probability p are allowed. A flea stays on its victim with probability $1-p$.

abilities. Interestingly, N fleas in the directed Ehrenfest next neighbor multiurn model by Kao and Luan [4] behave like one single flea that jumps with probability $p=1/N$. Consequently, for $p=1/N$, matrix (15) is equal to those in Eq. (7) in Ref. [4] specifying the transitions of the average number of fleas in the directed Ehrenfest next neighbor model. Figure 4 shows the fraction of the average number of fleas on
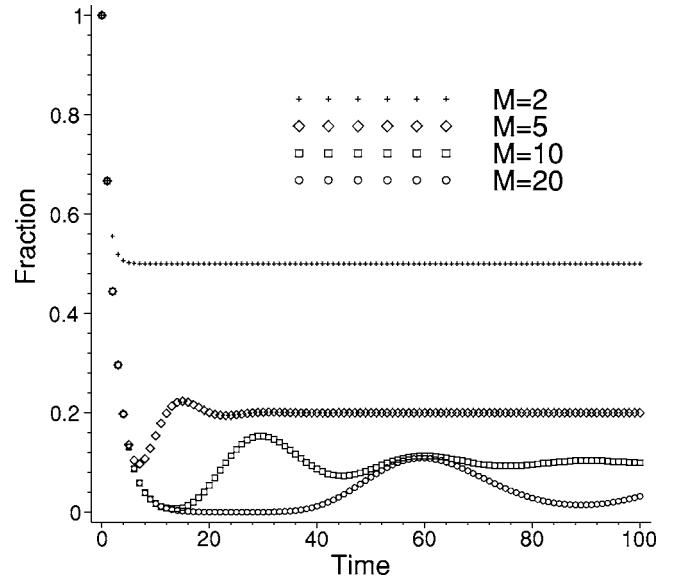

FIG. 5. Average number of fleas on dog $d=1$ as a function of time for $N=50$ fleas on $M=2, 5, 10$, and 20 dogs. The fraction, i.e., $\langle n \rangle_t / N$, is displayed for the case that all fleas were at time $t=0$ on dog $d=1$. In the model only jumps to the right neighbor dog with probability $p=1/3$ are allowed. A flea stays on its host with probability $1-p=2/3$.

dog 1 for different values of the jump probability p for the initial condition that all fleas sat at the beginning on dog $d = 1$. Strikingly, oscillations occur before the system reaches its equilibrium state. A second observation is that, as expected, the smaller the jump probability p the longer the transition time to the equilibrium state. Both observations are due to the circular arrangement of the dogs, that is, a flea returns to an individual dog after $t = M$ hops. Figure 5 displays the dependence on the system size M where the jump probability is fixed to $p = 1/3$. Similarly to the studies by Kao [4] oscillations occur for $M \geqslant 3$. Figure 6 reveals the time period $\tau = M/p$ of the oscillations. A flea peak wanders from dog to dog faster the higher is the value of p. Having passed M dogs the peak appears at the starting position. In Ref. [4] a suitable analytical approximation for the case $p = 1/N$ and the limit for small values of p is given.

VI. GENERAL DIRECTED MODEL

We now include jumps from dog j to the $(j+k)$th dog with probability p_k for $k > 1$. The eigenvalues of the system are given by

$$\mu_{j,\mathbf{p}} = p_0 + \sum_{k=1}^{M} p_k \exp\left(-\frac{2\pi i j k}{M}\right). \quad (19)$$

The corresponding eigenvectors are identical with those in Eq. (17). Consequently, we readily can write the single-flea probability distribution

$$\widetilde{Q}_{f,\mathbf{p}}^{t}(d) = \sum_{j=1}^{N} \mu_{j,\mathbf{p}}^{t} y_j^d y_{d_f}^{j*}. \quad (20)$$

VII. THE pq MODEL

Let us consider a simplification of the model which represents a generalization of the so-called pq model introduced by Kao [5]. It describes the simple case where all probabilities assumed to vanish except the first and the $(M-1)$th, i.e., $p_1 = p$, $p_{M-1} = q$. In contrast to Kao's model each flea can jump in a time step. Let us further assume $p_0 = 1 - p - q$. From Eq. (19) we read off the eigenvalues being

$$\mu_{j,p,q} = 1 + p(e^{2\pi i j/M} - 1) + q(e^{-2\pi i j/M} - 1). \quad (21)$$

Remember that the eigenvectors are independent of the choice of the probabilities p_k. Interestingly, for $p = q$, and $N = 1$, the model is equivalent to a random walker in a one-dimensional ring. For $p = q$ the model is equivalent to the undirected jump model and thus displays no oscillations. Figure 7 shows the fraction of the average number of fleas on dog $d = 1$ when initially all fleas start on it for various values of p and q with $p + q = \frac{1}{2}$. As expected, for $p \neq q$ oscillations occur.

CONCLUSION

To conclude, we introduced straightforward generalizations of three recently proposed multiurn models. For the models time discrete solutions were derived analytically. Strikingly, for the directed case we identified oscillations in the system as a consequence of the circle topology. Handy
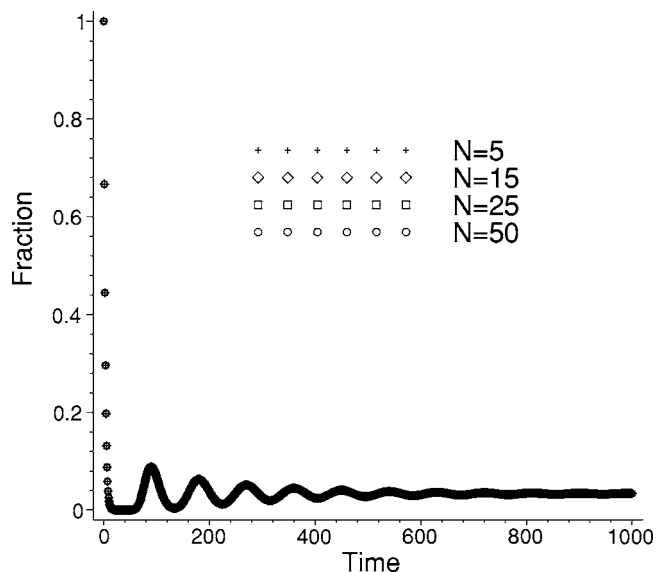

FIG. 6. Average number of fleas on the dog $d=1$ versus time for $N=5$, 15, 25, 50 fleas on $M=30$ dogs. The fraction, i.e., $\langle n \rangle_t / N$, is displayed for the case that all fleas were initially on dog $d=1$. Model and jump rate as in Fig. 5.

time continuous approximations as limiting cases of Ehrenfest models have been extensively discussed in the literature; see, e.g., Refs. [2,4,5,11] and references therein. The historical arbitrary restriction that only one flea jumps in a time step was dropped. As a consequence, the models better serve as basic models in the context of granular media (in
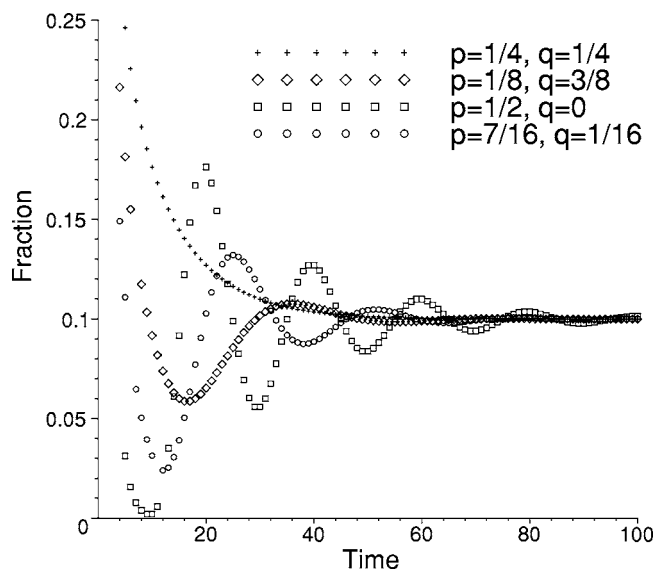

FIG. 7. The pq model. Fraction of the average number of fleas on dog $d=1$ versus time for $N=10$ fleas on $M=10$ dogs. The fraction, i.e., $\langle n \rangle_t / N$, is displayed assuming at time $t=0$ all fleas are on dog $d=1$. Here, only jumps to the right neighbor dog with probability p and jumps to the $(M-1)$th neighbor dog with probability q are allowed. A flea stays on its victim with probability $1-p-q$.

circularly arranged containers) than Ehrenfest models do. For instance, one may think of granular dynamics that can be modeled when the system's jump rate distribution $p(k)$ (k being the relative compartment position) is known.

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